

ON THE ENTANGLED ERGODIC THEOREM

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ABSTRACT. We study the convergence of the so-called entangled ergodic averages

$$\frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \dots A_1 T_1^{n_{\alpha(1)}},$$

where $k \leq m$ and $\alpha : \{1, \dots, m\} \rightarrow \{1, \dots, k\}$ is a surjective map. We show that, on general Banach spaces and without any restriction on the partition α , the above averages converge strongly as $N \rightarrow \infty$ under some quite weak compactness assumptions on the operators T_j and A_j . A formula for the limit based on the spectral analysis of the operators T_j and the continuous version of the result are presented as well.

1. INTRODUCTION

The classical mean ergodic theorem has inspired many mathematicians and led to several generalisations and extensions. We mention Berend, Lin, Rosenblatt, Tempelman [3] for modulated and subsequential ergodic theorems and e.g. Kra [13] for an overview on multiple ergodic theorems as well as for the history of the subjects and further references.

In this note we study a further extension of the mean ergodic theorem, namely the so-called entangled ergodic theorem. Let $\alpha : \{1, \dots, m\} \rightarrow \{1, \dots, k\}$ be a surjective map for some positive integers $k \leq m$, and take T_1, \dots, T_m and A_1, \dots, A_{m-1} to be linear operators on a Banach space X . We investigate the convergence of the entangled Cesàro means

$$(1) \quad \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \dots A_1 T_1^{n_{\alpha(1)}}.$$

This type of ergodic theorems was introduced by Accardi, Hashimoto, Obata [1] motivated by quantum stochastics and was then studied by Liebscher [15] and F. Fidaleo [9, 11]. In their studies, $T_1 = \dots = T_m =: U$ is a unitary operator on a Hilbert space. Besides a technical assumption of Liebscher [15], there were two basic situations in which weak or strong convergence of the entangled ergodic averages could be proved. The first is when A_j are arbitrary and the unitary operator U is almost periodic (see Definition 2.1 below), see Liebscher [15] and

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Fidaleo [11]. The second is when the operators A_j are compact and U is arbitrary unitary, see Fidaleo [9].

In this paper we consider a more general situation. We assume the operators T_j belong to a large class including power bounded operators on reflexive Banach spaces. Further, we require a quite general compactness condition on the pairs (A_j, T_j) generalising both of the above cases. More precisely, we make the following assumptions.

(A1) (Weakly compact orbits of T_j)

The operator T_m is power bounded and totally ergodic and each of T_1, \dots, T_{m-1} has relatively weakly compact orbits, i.e., $\{T_j^n x : n \in \mathbb{N}\}$ is relatively compact in X in the weak topology for every $x \in X$ and $1 \leq j \leq m-1$.

(A2) (Joint compactness of (A_j, T_j))

Every A_j is compact on the orbits of T_j , i.e., $\{A_j T_j^n x : n \in \mathbb{N}\}$ is relatively compact in X for every $x \in X$ and $1 \leq j \leq m-1$.

Recall that an operator T is called totally ergodic if the operators λT are mean ergodic for every $\lambda \in \mathbb{T}$, \mathbb{T} the unit circle. Every operator with relatively weakly compact orbits is automatically totally ergodic. Note that assumption (A1) is not very restrictive. As mentioned above, every power bounded operator on a reflexive Banach space has relatively weakly compact orbits by the Banach–Alaoglu theorem. Another important class of examples is given by power bounded positive operators on a Banach lattice $L^1(\mu)$ preserving an order interval generated by a strictly positive function, see e.g. Schaefer [18, Thm. II.5.10(f) and Prop. II.8.3] or [7, Section I.1] for further information. While forming a large class, operators with relatively weakly compact orbits admit good asymptotic properties, see Theorem 2 below.

Under the assumptions (A1) and (A2) we show that the entangled Cesàro means (1) converge strongly and describe their limit operator, see Theorem 3. It turns out that only specially interacting projections corresponding to unimodular eigenvalues of T_j (in combination with the operators A_j) contribute to the limit.

The paper is organised as follows. We first treat the special case assuming that all but the last T_j are almost periodic (Section 2). Here we show how to reduce the problem to the case when $T_1 = \dots = T_{m-1}$ and $A_1 = \dots = A_{m-1}$. In the context of Hilbert spaces and pair partitions, this case was considered recently in Fidaleo [11]. In Section 3 the general case is treated. We further discuss the connection to non-commutative multiple ergodic theorems, them being an important recent field of research. In Section 5, we finally study the case of strongly continuous semigroups and the strong convergence of the ergodic averages

$$\frac{1}{t^k} \int_{[0,t]^k} T_m(s_{\alpha(m)}) A_{m-1} T_{m-1}(s_{\alpha(m-1)}) A_{m-2} \dots A_1 T_1(s_{\alpha(1)}) ds_1 \dots ds_k.$$

Note that the study of convergence of entangled ergodic averages for the continuous time scale seems not be new.

Before proceeding we first show by an example that one cannot drop assumption (A2) even when $m = 2$, the operator $T_1 = T_2 =: U$ is unitary and weakly stable, i.e., satisfies $\lim_{n \rightarrow \infty} U^n = 0$ in the weak operator topology.

Proposition 1. *There exists a Hilbert space H , a weakly stable unitary operator U on H and an operator $A \in \mathcal{L}(H)$ such that the Cesàro means*

$$\frac{1}{N} \sum_{n=1}^N U^n A U^n$$

do not converge weakly.

Proof. Let $H := \ell^2(\mathbb{Z})$ and consider the standard orthonormal base $\{e_b\}_{b \in \mathbb{Z}}$. Let then U be the unitary left shift operator. Let further $(f(n))_{n \in \mathbb{N}^+}$ be a 0–1 sequence that is not Cesàro-summable, and define A by

$$Ae_b := \begin{cases} e_{f(-b)-b} & \text{whenever } b < 0, \\ e_b & \text{whenever } b \geq 0. \end{cases}$$

Then A is a bounded operator with $\|A\| = \sqrt{3}$, and we have $U^n A U^n e_0 = e_{f(n)}$ for all $n \in \mathbb{N}$. Hence $\langle U^n A U^n e_0, e_0 \rangle = 1 - f(n)$, and since $(f(n))_{n \in \mathbb{N}^+}$ is not Cesàro-summable, the Cesàro-means $\frac{1}{N} \sum_{n=1}^N U^n A U^n$ cannot converge weakly either. \square

2. THE ALMOST PERIODIC CASE

We first consider the situation when all operators T_j are almost periodic. In the case when α is a pair partition, X is a Hilbert space and the operators T_j are all equal to a unitary operator, this has been studied by Liebscher [15] and Fidaleo [11].

Definition 2.1. An operator $T \in \mathcal{L}(X)$ acting on a Banach space is called *almost periodic* if it is power bounded (i.e. $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$) and satisfies

$$X = \overline{\text{lin}} \{x \in X \mid \exists \lambda \in \mathbb{T} : Tx = \lambda x\}.$$

The following result is the first step towards the general situation.

Theorem 1. *Let X be a Banach space, T_1, \dots, T_{m-1} be almost periodic operators on X , $T_m \in \mathcal{L}(X)$ be power bounded and totally ergodic, $A_1, \dots, A_{m-1} \in \mathcal{L}(X)$, and $\alpha : \{1, \dots, m\} \rightarrow \{1, \dots, k\}$ be surjective for some $k \leq m$. Then the entangled Cesàro means*

$$(2) \quad \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \dots A_1 T_1^{n_{\alpha(1)}}$$

converge strongly as $N \rightarrow \infty$.

Proof. We shall proceed by induction on m , giving an explicit form for the limit.

First recall that any operator with relatively weakly compact orbits (in particular any almost periodic operator) is mean ergodic, see e.g. [7, Thm. I.2.9], and to have relatively weakly compact orbits is an invariant property under multiplication by a unimodular constant. We may therefore introduce the mean ergodic projections

$$P_\lambda^{(j)} := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N (\lambda^{-1} T_j)^n,$$

where $1 \leq j \leq m$ and $|\lambda| = 1$.

We now show that the limit of the entangled Cesàro means (2) is given by the formal sum

$$(3) \quad \sum_{\substack{\lambda_j \in \sigma_j \ (1 \leq j \leq m) \\ \prod_{i \in \alpha^{-1}(a)} \lambda_i = 1 \ (1 \leq a \leq k)}} P_{\lambda_m}^{(m)} A_{m-1} P_{\lambda_{m-1}}^{(m-1)} A_{m-2} \dots A_1 P_{\lambda_1}^{(1)}.$$

Here, σ_j denotes the point spectrum of T_j ($1 \leq j \leq m$) and (3) should be understood as the strong limit of the net

$$\left\{ \sum_{\substack{\lambda_j \in F_j \ (1 \leq j \leq m) \\ \prod_{i \in \alpha^{-1}(a)} \lambda_i = 1 \ (1 \leq a \leq k)}} P_{\lambda_m}^{(m)} A_{m-1} P_{\lambda_{m-1}}^{(m-1)} A_{m-2} \dots A_1 P_{\lambda_1}^{(1)} \right\}_{F_j \subset \sigma_j \text{ finite } (1 \leq j \leq m)}.$$

As seen above, this holds for $m = 1$. Suppose we know that it holds for any choice of $(m-2)$ almost periodic operators T_j and $(m-2)$ bounded operators A_i ($m \geq 2$). We may suppose without loss of generality that $\alpha(1) = 1$.

If now $\alpha^{-1}(1) = \{1\}$, then (2) can be written as

$$\left(\frac{1}{N^{(k-1)}} \sum_{n_2, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \dots A_2 T_2^{n_{\alpha(2)}} \right) A_1 \left(\frac{1}{N} \sum_{n_1=1}^N T_1^{n_1} \right).$$

By the induction hypotheses and the joint continuity of multiplication in the strong operator topology, this expression converges to

$$\begin{aligned} & \left(\sum_{\substack{\lambda_j \in \sigma_j \ (2 \leq j \leq m) \\ \prod_{i \in \alpha^{-1}(a)} \lambda_i = 1 \ (2 \leq a \leq k)}} P_{\lambda_m}^{(m)} A_{m-1} P_{\lambda_{m-1}}^{(m-1)} A_{m-2} \dots A_2 P_{\lambda_2}^{(2)} \right) A_1 P_1^{(1)} \\ &= \sum_{\substack{\lambda_j \in \sigma_j \ (1 \leq j \leq m) \\ \prod_{i \in \alpha^{-1}(a)} \lambda_i = 1 \ (1 \leq a \leq k)}} P_{\lambda_m}^{(m)} A_{m-1} P_{\lambda_{m-1}}^{(m-1)} A_{m-2} \dots A_1 P_{\lambda_1}^{(1)}. \end{aligned}$$

If, on the other hand, there exists an $1 < l \leq m$ with $\alpha(l) = 1$, then consider an eigenvector $x \in X$ of T_1 pertaining to some eigenvalue $\lambda \in \mathbb{T}$. We can rewrite the entangled means applied to x as

$$\begin{aligned} & \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \dots A_1 T_1^{n_{\alpha(1)}} x \\ &= \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \dots A_1 \lambda^{n_{\alpha(1)}} x \\ &= \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \dots (\lambda T_l)^{n_{\alpha(l)}} \dots A_2 T_2^{n_{\alpha(2)}} (A_1 x). \end{aligned}$$

This reduces the problem to the $(m-2)$ almost periodic operators $T_2, \dots, \lambda T_l, \dots, T_m$, and $(m-2)$ bounded operators A_2, \dots, A_{m-1} . The induction hypotheses together with $x = P_\lambda^{(1)} x$ yields that the limit is

$$\begin{aligned}
& \sum_{\substack{\lambda_j \in \sigma_j \ (2 \leq j \leq m, j \neq l) \\ \lambda_l \in \lambda \sigma_l \\ \prod_{i \in \alpha^{-1}(a) \setminus \{1\}} \lambda_i = 1 \ (1 \leq a \leq k)}} P_{\lambda_m}^{(m)} A_{m-1} P_{\lambda_{m-1}}^{(m-1)} A_{m-2} \dots P_{\lambda^{-1} \lambda_l}^{(l)} \dots A_2 P_{\lambda_2}^{(2)} (A_1 x) \\
&= \sum_{\substack{\lambda_j \in \sigma_j \ (2 \leq j \leq m) \\ \lambda_1 = \lambda \\ \prod_{i \in \alpha^{-1}(a)} \lambda_i = 1 \ (1 \leq a \leq k)}} P_{\lambda_m}^{(m)} A_{m-1} P_{\lambda_{m-1}}^{(m-1)} A_{m-2} \dots P_{\lambda_l}^{(l)} \dots A_2 P_{\lambda_2}^{(2)} A_1 P_{\lambda_1}^{(1)} x \\
&= \sum_{\substack{\lambda_j \in \sigma_j \ (1 \leq j \leq m) \\ \prod_{i \in \alpha^{-1}(a)} \lambda_i = 1 \ (1 \leq a \leq k)}} P_{\lambda_m}^{(m)} A_{m-1} P_{\lambda_{m-1}}^{(m-1)} A_{m-2} \dots A_1 P_{\lambda_1}^{(1)} x.
\end{aligned}$$

Since the eigenvectors of T_1 corresponding to unimodular eigenvalues span X and the entangled Cesàro means are uniformly bounded, we obtain the convergence on X to the required limit. \square

It is interesting that in Theorem 1 it is not important whether all operators T_j ($1 \leq j \leq m-1$) are different or equal, and the same for the operators A_j , as the following shows. We shall need this fact in the proof of Theorem 3 below.

Proposition 2. *Suppose we know that Theorem 1 and the form of the limit given by (3) hold under additional assumption $T_1 = \dots = T_{m-1}$ and $A_1 = \dots = A_{m-1}$. Then it also holds in full generality.*

Proof. Consider the space $\mathcal{X} := X^m$ with the diagonal operators

$$\begin{aligned}
\mathcal{T} &:= \text{diag}(T_1, T_2, \dots, T_{m-1}, I) \in \mathcal{L}(\mathcal{X}), \\
\mathcal{S} &:= \text{diag}(I, I, \dots, I, T_m) \in \mathcal{L}(\mathcal{X})
\end{aligned}$$

and the off-diagonal operator

$$\mathcal{A} := ((\delta_{a-1,b} A_a))_{a,b} \in \mathcal{L}(\mathcal{X}),$$

where δ is the Kronecker symbol. Since all the T_j 's ($1 \leq j \leq m-1$) are almost periodic, so is \mathcal{T} , and \mathcal{S} is clearly totally ergodic on \mathcal{X} . We can hence apply the weaker statement of Theorem 1 to the operators \mathcal{T} , \mathcal{S} and \mathcal{A} . For the vector $(x, 0, \dots, 0)^T \in \mathcal{X}$ for some $x \in X$, this yields the existence of

$$(4) \quad \lim_{N \rightarrow \infty} \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N \mathcal{S}^{n_{\alpha(m)}} \mathcal{A} \mathcal{T}^{n_{\alpha(m-1)}} \mathcal{A} \dots \mathcal{A} \mathcal{T}^{n_{\alpha(1)}} (x, 0, \dots, 0)^T.$$

However, each of the summands has the form

$$(0, \dots, 0, T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \dots A_1 T_1^{n_{\alpha(1)}} x)^T,$$

and hence the convergence of (4) in the last coordinate implies the required convergence of (2). Concerning the explicit form of the limit in question, the expression (3) yields that it is the last coordinate of

$$(5) \quad \sum_{\substack{\lambda_j \in \sigma \ (1 \leq j \leq m) \\ \prod_{i \in \alpha^{-1}(a)} \lambda_i = 1 \ (1 \leq a \leq k)}} \mathcal{P}_{\lambda_m}^{\mathcal{S}} \mathcal{A} \mathcal{P}_{\lambda_{m-1}}^{\mathcal{T}} \dots \mathcal{A} \mathcal{P}_{\lambda_1}^{\mathcal{T}}(x, 0, \dots, 0)^T,$$

where $\mathcal{P}_{\lambda}^{\mathcal{S}}$ and $\mathcal{P}_{\lambda}^{\mathcal{T}}$ are the mean ergodic projections onto the eigenspace corresponding to λ of \mathcal{S} and \mathcal{T} , respectively. Due to the diagonality of \mathcal{T} and \mathcal{S} , each of the components in $\mathcal{X} = X^m$ is \mathcal{T} - and \mathcal{S} -invariant, hence we in fact have $\mathcal{P}_{\lambda}^{\mathcal{T}} = \text{diag}(P_{\lambda}^{(1)}, \dots, P_{\lambda}^{(m-1)}, \mathbb{1}_{\{1\}}(\lambda)I)$ and $\mathcal{P}_{\lambda}^{\mathcal{S}} = \text{diag}(\mathbb{1}_{\{1\}}(\lambda)I, \dots, \mathbb{1}_{\{1\}}(\lambda)I, P_{\lambda}^{(m)})$ where $\mathbb{1}_M$ denotes the characteristic function of a set M . The summands in (5) thus have the form

$$(0, \dots, 0, P_{\lambda_m}^{(m)} A_{m-1} P_{\lambda_{m-1}}^{(m-1)} A_{m-2} \dots A_1 P_{\lambda_1}^{(1)} x)^T.$$

Taking into consideration that the mean ergodic projections satisfy $\text{Ran } P_{\lambda}^{(j)} = \{y \in X \mid T_j y = \lambda y\}$, and hence $P_{\lambda}^{(j)} = 0$ whenever $\lambda \notin \sigma_j$, the limit reduces to the required form. \square

3. THE GENERAL CASE

We now extend the results from the previous section to the case when only the orbits along the family of operators $\{A_i T_i^n\}_{n \in \mathbb{N}^+}$ are relatively compact for all $1 \leq i \leq m-1$, i.e., assuming (A2).

The key for our considerations will be the following extended version of a classical decomposition theorem, see e.g. Krengel [14, Section 2.2.4] or [7, Theorem II.4.8].

Theorem 2. (Jacobs–Glicksberg–de Leeuw decomposition) *Let X be a Banach space and let $T \in \mathcal{L}(X)$ have relatively weakly compact orbits. Then $X = X_r \oplus X_s$, where*

$$\begin{aligned} X_r &:= \overline{\text{lin}} \{x \in X \mid Tx = \lambda x \text{ for some } \lambda \in \mathbb{T}\}, \\ X_s &:= \left\{ x \in X \mid \lim_{j \rightarrow \infty} T^{n_j} x = 0 \text{ weakly for some sequence } \{n_j\}_{j=1}^{\infty} \text{ with density 1} \right\}, \end{aligned}$$

with both subspaces being invariant under T . In addition, if X' is separable, then there exists a sequence $\{n_j\}_{j=1}^{\infty}$ with density 1 such that $\lim_{j \rightarrow \infty} T^{n_j}|_{X_s} = 0$ weakly.

Recall that the density of a set $M \subset \mathbb{N}$ is defined by

$$d(M) = \lim_{n \rightarrow \infty} \frac{|M \cap \{1, \dots, n\}|}{n} \leq 1,$$

whenever the above limit exists.

We further need the following well-known lemmas.

Lemma 1. (Koopman–von Neumann) *For a bounded sequence $\{a_n\}_{n=1}^{\infty} \subset [0, \infty)$ the following assertions are equivalent.*

$$(a) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n a_k = 0.$$

- (b) *There exists a subsequence $\{n_j\}_{j=1}^\infty$ of \mathbb{N} with density 1 such that $\lim_{j \rightarrow \infty} a_{n_j} = 0$.*

We refer to e.g. Petersen [17, p. 65] for the proof.

Lemma 2. *Let X be a Banach space and let $\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty \subset \mathcal{L}(X)$. Then the following assertion holds. If both $\{T_n x : n \in \mathbb{N}\}$ and $\{S_n x : n \in \mathbb{N}\}$ are relatively compact in X for every $x \in X$, then so is $\{T_n S_n x : n \in \mathbb{N}\}$ for every $x \in X$.*

Proof. Since compact sets are bounded and by the uniform boundedness principle, there exists $M > 0$ such that $\|T_n\| \leq M$ and $\|S_n\| \leq M$ holds for every $n \in \mathbb{N}^+$. Take now $x \in X$ and a sequence $\{n_j\} \subset \mathbb{N}$. Then there exists a subsequence $\{m_j\}$ of $\{n_j\}$ such that $\lim_{j \rightarrow \infty} S_{m_j} x = y$ for some $y \in X$. Furthermore, there exists a subsequence of $\{m_j\}$ which we again denote by $\{m_j\}$ such that $\lim_{j \rightarrow \infty} T_{m_j} y = z$ for some $z \in X$. This yields

$$\|T_{m_j} S_{m_j} x - z\| \leq M \|S_{m_j} x - y\| + \|T_{m_j} y - z\| \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

proving relative compactness of $\{T_n S_n x : n \in \mathbb{N}\}$. \square

The following is the main result of the paper.

Theorem 3. *Let X be a Banach space and $\alpha : \{1, \dots, m\} \rightarrow \{1, \dots, k\}$ be surjective for some $k \leq m$. Let further $T_1, \dots, T_m, A_1, \dots, A_{m-1} \in \mathcal{L}(X)$ satisfy assumptions (A1) and (A2). Then the entangled ergodic averages*

$$\frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \dots A_1 T_1^{n_{\alpha(1)}}$$

converge strongly, and their limit is given by

$$\sum_{\substack{\lambda_j \in \sigma_j \ (1 \leq j \leq m) \\ \prod_{i \in \alpha^{-1}(a)} \lambda_i = 1 \ (1 \leq a \leq k)}} P_{\lambda_m}^{(m)} A_{m-1} P_{\lambda_{m-1}}^{(m-1)} A_{m-2} \dots A_1 P_{\lambda_1}^{(1)},$$

where $\sigma_j = P_\sigma(T_j) \cap \mathbb{T}$ and $P_{\lambda_j}^{(j)}$ is the projection onto the eigenspace of T_j corresponding to λ_j , i.e., the mean ergodic projection of the operator $\overline{\lambda_j} T_j$.

Proof. As in the proof of Corollary 2 we may assume $T_j = T$ and $A_j = A$ for some $T, A \in \mathcal{L}(X)$ and all $1 \leq j \leq m-1$. For $x \in X$, we have to show convergence of

$$(6) \quad \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A T^{n_{\alpha(m-1)}} A \dots A T^{n_{\alpha(1)}} x.$$

By Theorem 2, the summands in (6) satisfy

$$\begin{aligned} T_m^{n_{\alpha(m)}} A T^{n_{\alpha(m-1)}} A \dots A T^{n_{\alpha(1)}} x \\ &= \sum_{a=1}^{m-1} T_m^{n_{\alpha(m)}} A \dots A T^{n_{\alpha(a)}} P_s A T^{n_{\alpha(a-1)}} P_r A \dots A T^{n_{\alpha(1)}} P_r x \\ &+ T_m^{n_{\alpha(m)}} A T^{n_{\alpha(m-1)}} P_r A \dots A T^{n_{\alpha(1)}} P_r x, \end{aligned}$$

where P_r and P_s are the projections onto X_r and X_s pertaining to T from Theorem 2, respectively. By Theorem 1, the averages of the second summand above converge

to the desired limit. It remains to show that the averages of the first summand converge to 0, i.e., that for every $x \in X$ and $1 \leq a \leq m-1$ one has

$$(7) \quad \lim_{N \rightarrow \infty} \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N T_m^{n_{\alpha(m)}} A T^{n_{\alpha(m-1)}} A \dots A T^{n_{\alpha(a)}} P_s A T^{n_{\alpha(a-1)}} P_r A \dots A T^{n_{\alpha(1)}} P_r x = 0.$$

Consider

$$K := \{A T^{n_{a-1}} P_r A \dots A T^{n_1} P_r x \mid n_{a-1}, \dots, n_1 \in \mathbb{N}\}$$

which is relatively compact by assumption and Lemma 2(a). We now show that the dual space of the smallest T -invariant subspace Y containing K is separable. Observe first that $Y = \overline{\text{lin}}\{T^n x \mid n \in \mathbb{N}, x \in K\}$. We first show that the set $\text{Orb}(K) := \{T^n x \mid n \in \mathbb{N}, x \in K\}$ is relatively weakly compact. Take a sequence $\{T^{n_j} x_j\}_{j=1}^\infty$ with $x_j \in K$ and $n_j \in \mathbb{N}$. Since K is relatively compact, there exists a subsequence of $\{x_j\}$ (which we again denote by $\{x_j\}$) converging to some z . Moreover, since T has relatively weakly compact orbits, there is a subsequence of $\{n_j\}$ (which we again denote by $\{n_j\}$) such that $\lim_{j \rightarrow \infty} T^{n_j} z = w$ weakly for some $w \in X$. So we have

$$|\langle T^{n_j} x_j - w, y \rangle| \leq |\langle T^{n_j} x_j - T^{n_j} z, y \rangle| + |\langle T^{n_j} z - w, y \rangle| \rightarrow 0 \quad \forall y \in X',$$

i.e., $\lim_{j \rightarrow \infty} T^{n_j} x_j = w$ weakly, and therefore $\text{Orb}(K)$ is relatively weakly compact. Since Y is separable, and the weak topology is metrisable on weakly compact subsets of separable spaces (see e.g. Dunford, Schwartz [5, Theorem V.6.3]), the weak topology on $\text{Orb}(K)$ is metrisable. So it is induced by countably many $\{y_n\} \subset Y'$, implying separability of Y' by $Y = \overline{\text{lin}}\text{Orb}(K)$.

Theorem 2 assures now the existence of a sequence $\{n_j\} \subset \mathbb{N}$ with density 1 such that $\lim_{j \rightarrow \infty} T^{n_j} P_s y = 0$ weakly for every $y \in K$. This implies

$$(8) \quad \lim_{j \rightarrow \infty} A T^{n_j} P_s y = 0 \text{ weakly for every } y \in K.$$

We now show that

$$(9) \quad \lim_{j \rightarrow \infty} \|A T^{n_j} P_s y\| = 0 \text{ uniformly in } y \in K.$$

Indeed, assume that for some $y \in K$, $A T^{n_j} P_s y$ does not converge strongly to zero. Then there exist $\delta > 0$ and a subsequence $\{m_j\}$ of $\{n_j\}$ such that $\|A T^{m_j} P_s y\| \geq \delta$ for every j . By relative compactness of $\{A T^n P_s y : n \in \mathbb{N}\}$, the sequence $\{A T^{m_j} P_s y\}_{j=1}^\infty$ has a strong accumulation point which by (8) must be zero, a contradiction. Thus, $\lim_{j \rightarrow \infty} \|A T^{n_j} P_s y\| = 0$ for every $y \in K$ and therefore uniform in $y \in K$, since strong convergence in $\mathcal{L}(X)$ implies uniform strong convergence on compact sets. Since the sequence $\{n_j\}$ has density 1, the equation (7) follows from (9) and Lemma 1. \square

4. CONNECTION TO CONVERGENCE OF MULTIPLE ERGODIC AVERAGES

In this section we discuss connection between the considered above entangled ergodic theorems and the important topic as multiple ergodic averages. The latter is concerned with non-commutative dynamical systems and was studied by

Niculescu, Ströh and Zsidó [16], Duvenhage [6], Beyers, Duvenhage and Ströh [4], Fidaleo [10], and Austin, Eisner, Tao [2].

We first introduce what we mean by a non-commutative dynamical system and corresponding multiple ergodic averages.

Definition 4.1. A von Neumann (or non-commutative) dynamical system is a triple $(\mathcal{M}, \tau, \beta)$, where \mathcal{M} is a von Neumann algebra, $\tau : \mathcal{M} \rightarrow \mathbb{C}$ is a faithful normal trace, and $\beta : \mathcal{M} \rightarrow \mathcal{M}$ is a τ -preserving $*$ -automorphism. We say for $k \in \mathbb{N}$ that the multiple ergodic averages

$$(10) \quad \frac{1}{N} \sum_{n=1}^N \beta^n(a_1) \beta^{2n}(a_2) \dots \beta^{kn}(a_k)$$

converge *strongly* if they converge in the τ -norm defined by $\|a\|_\tau := \sqrt{\tau(aa^*)}$. The averages in (10) are called *weakly* convergent if

$$\frac{1}{N} \sum_{n=1}^N \tau(a_0 \beta^n(a_1) \beta^{2n}(a_2) \dots \beta^{kn}(a_k))$$

converges as $N \rightarrow \infty$ for every $a_0 \in \mathcal{M}$.

We recall that by the Gel'fand–Neumark–Segal theory, \mathcal{M} can be identified with a dense subspace of a Hilbert space, where the Hilbert space can be obtained as the completion of \mathcal{M} with respect to the τ -norm. Thus, identifying elements of \mathcal{M} with elements in H and by the standard density argument, strong convergence of the multiple ergodic averages (10) corresponds to norm convergence in H and weak convergence of (10) corresponds to weak convergence in H .

Recall further that for the automorphism β there exists a unitary operator $u \in \mathcal{L}(H)$ such that $\beta(a) = uau^{-1}$, see e.g. [12, Prop. 4.5.3]. (Note that u does not necessarily belong to \mathcal{M} .) Thus, averages (10) take the form

$$(11) \quad \frac{1}{N} \sum_{n=1}^N u^n a_1 u^n a_2 \dots u^n a_k u^{-kn},$$

i.e., are a special case of entangled ergodic averages for the constant partition $\alpha(j) = 1$ for every $j \in 1, \dots, k$ and the operators u, \dots, u, u^{-k} .

It is well-known that strong (weak) topology and strong (weak) operator topology on \mathcal{M} coincide. Therefore, there is a direct correspondence between strong (weak) convergence of multiple ergodic averages (10) and strong (weak) *operator* convergence of the entangled ergodic averages (11), cf. also Fidaleo [10].

Proposition 3. *Let $(\mathcal{M}, \tau, \beta)$ be a von Neumann dynamical system and H and u as above. Let further $a_1, \dots, a_k \in \mathcal{M}$. Then the multiple ergodic averages (10) converge strongly (weakly) if and only if the entangled averages (11) converge in the strong (weak) operator topology.*

As was shown in [2], multiple ergodic averages (10) do not converge in general for $k \geq 3$. Theorem 3 shows now that for every von Neumann dynamical system there is a class \mathcal{K} depending on the system such that the multiple ergodic averages converge strongly whenever $a_1, \dots, a_k \in \mathcal{K}$, and \mathcal{K} can be chosen as the subspace

of all elements $a \in \mathcal{M}$ such that $\{au^n : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{L}(H)$ for the strong operator topology.

Remark. Note that also more general sequences of powers than arithmetic sequences for the multiple ergodic averages can be treated by a slight modification in the last inductive step of the proof of Theorem 1. For instance, consider the averages

$$(12) \quad \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N \beta^{n_{\alpha(1)}}(a_1) \beta^{n_{\alpha(1)} + \alpha(2)}(a_2) \dots \beta^{\sum_{j=1}^m n_{\alpha(j)}}(a_m).$$

These can then be rewritten as

$$(13) \quad \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N u^{n_{\alpha(1)}} a_1 u^{\alpha(2)} a_2 \dots u^{n_{\alpha(m)}} a_m u^{-\sum_{j=1}^m n_{\alpha(j)}}.$$

The last exponent being a sum of $\alpha(j)$'s rather than a single one does not matter on the almost weakly stable part, the compactness arguments of the proof of Theorem 3 still hold. On the almost periodic part however, suppose x is an eigenvector to the unimodular eigenvalue λ . Then

$$\begin{aligned} & \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N u^{n_{\alpha(1)}} a_1 u^{\alpha(2)} a_2 \dots u^{n_{\alpha(m)}} a_m u^{-\sum_{j=1}^m n_{\alpha(j)}} x \\ &= \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N u^{n_{\alpha(1)}} a_1 u^{\alpha(2)} a_2 \dots u^{n_{\alpha(m)}} a_m \lambda^{-\sum_{j=1}^m n_{\alpha(j)}} x, \end{aligned}$$

and the powers of the eigenvalue λ have to be pulled forward to not only a single operator as in the inductive proof of Theorem 1, but distributed amongst all of them to get back to the standard form:

$$\begin{aligned} & \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N u^{n_{\alpha(1)}} a_1 u^{\alpha(2)} a_2 \dots u^{n_{\alpha(m)}} a_m \lambda^{-\sum_{j=1}^m n_{\alpha(j)}} x \\ &= \frac{1}{N^k} \sum_{n_1, \dots, n_k=1}^N (\lambda^{-1} u)^{n_{\alpha(1)}} a_1 (\lambda^{-1} u)^{\alpha(2)} a_2 \dots (\lambda^{-1} u)^{n_{\alpha(m)}} a_m x. \end{aligned}$$

Hence the averages (12) converge strongly if $\{a_j u^n : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{L}(H)$ for the strong operator topology for every $1 \leq j \leq m$.

5. CONTINUOUS CASE

In this section we treat the continuous time scale, where the operators T_j and their powers are replaced by strongly continuous (C_0-) semigroups $(T_j(t))_{t \geq 0}$. The study of the continuous version for entangled ergodic averages seems to be new. Some steps in the proofs are similar to the discrete case and will be skipped. For the general theory of strongly continuous semigroups we refer to e.g. Engel, Nagel [8]. For a semigroup $(T(t))_{t \geq 0}$ we often write $T(\cdot)$.

Definition 5.1. A C_0 -semigroup of operators $(T(t))_{t \geq 0} \subset \mathcal{L}(X)$ acting on a Banach space X is called *almost periodic* if it is bounded (i.e. $\sup_{t \geq 0} \|T(t)\| < \infty$) and satisfies

$$X = \overline{\lim} \{x \in X \mid \exists \varphi \in \mathbb{R} : T(t)x = e^{i\varphi t}x \forall t \geq 0\}.$$

Recall that by the spectral mapping theorem (see e.g. [8, Corollary IV.3.8]), $T(t)x = e^{i\varphi t}x$ for every $t \geq 0$ if and only if $Bx = i\varphi x$ for the generator B of $T(\cdot)$.

Theorem 4. Let X be a Banach space, $T_1(\cdot), \dots, T_{m-1}(\cdot)$ be almost periodic C_0 -semigroups on X , $T_m(\cdot)$ a bounded totally ergodic C_0 -semigroup on X and $A_1, \dots, A_{m-1} \in \mathcal{L}(X)$. Then the entangled Cesàro means

$$(14) \quad \frac{1}{t^k} \int_{[0,t]^k} T_m(s_{\alpha(m)}) A_{m-1} T_{m-1}(s_{\alpha(m-1)}) A_{m-2} \dots A_1 T_1(s_{\alpha(1)}) ds_1 \dots ds_k$$

converge strongly as $t \rightarrow \infty$.

The integrals in (14) are defined strongly. Recall that a semigroup $(T(t))_{t \geq 0}$ is called totally ergodic if the semigroup $(e^{i\varphi t} T(t))_{t \geq 0}$ is mean ergodic for every $\varphi \in \mathbb{R}$.

Proof. Since almost periodic C_0 -semigroups are totally ergodic by the standard density argument, we can for each $T_j(\cdot)$ ($1 \leq j \leq m$) define the projections $P_\varphi^{(j)}$ as the mean ergodic projections of $(e^{-i\varphi t} T_j(t))_{t \geq 0}$ with range $\{y \in X \mid T_j(t)y = e^{i\varphi t}y \forall t \geq 0\}$. Let σ_j denote the point spectrum of the generator B_j of the semigroup $T_j(\cdot)$ on $i\mathbb{R}$ ($1 \leq j \leq m$).

The proof can then be concluded by an induction argument on m analogous to the discrete case showing that the limit of the entangled Cesàro means (14) is given by

$$(15) \quad \sum_{\substack{\varphi_j \in \sigma_j \ (1 \leq j \leq m) \\ \sum_{i \in \alpha^{-1}(a)} \varphi_i = 0 \ (1 \leq a \leq k)}} P_{\varphi_m}^{(m)} A_{m-1} P_{\varphi_{m-1}}^{(m-1)} A_{m-2} \dots A_1 P_{\varphi_1}^{(1)}.$$

□

We shall use the following analogue of Theorem 2, see e.g. [7, Theorem III.5.7].

Theorem 5. (Continuous Jacobs–Glicksberg–de Leeuw decomposition) Let X be a Banach space and let $T(\cdot) \subset \mathcal{L}(X)$ be a C_0 -semigroup with relatively weakly compact orbits, i.e., such that $\{T(t)x : t \in [0, \infty)\}$ is relatively compact in X in the weak topology for every $x \in X$. Then $X = X_r \oplus X_s$, where

$$\begin{aligned} X_r &= \overline{\lim} \{x \in X \mid T(t)x = e^{i\varphi t}x \forall t \geq 0 \text{ for some } \varphi \in \mathbb{R}\}, \\ X_s &= \left\{x \in X \mid \lim_{M \ni j \rightarrow \infty} T(t)x = 0 \text{ weakly for some } M \subset [0, \infty) \text{ with density } 1\right\}, \end{aligned}$$

with both subspaces being invariant under T . In addition, if X' is separable, then there exists a set $M \subset [0, \infty)$ with density 1 such that $\lim_{M \ni t \rightarrow \infty} T(t)|_{X_s} = 0$ weakly.

The density of a set $M \subset [0, \infty)$ is defined by

$$d(M) = \lim_{t \rightarrow \infty} \frac{\lambda(M \cap [0, t])}{t} \leq 1,$$

with $\lambda(\cdot)$ denoting the Lebesgue measure, whenever the above limit exists.

We further need the following continuous version of Lemma 1.

Lemma 3. (Koopman–von Neumann, continuous version) *For a continuous function $f : [0, \infty) \rightarrow [0, \infty)$ the following assertions are equivalent.*

- (a) $\lim_{t \rightarrow \infty} \frac{1}{t} \int_{[0, t]} f(s) ds = 0.$
- (b) *There exists a subset M of $[0, \infty)$ with density 1 such that $\lim_{s \in M, s \rightarrow \infty} f(s) = 0.$*

For the proof, which is analogous to the discrete case, see e.g. [7, Lemma III.5.2].

The main result of this section is the following.

Theorem 6. *Let X be a Banach space, $m \in \mathbb{N}$, $T_m(\cdot), \dots, T_1(\cdot)$ be C_0 -semigroups on X , $A_1, \dots, A_{m-1} \in \mathcal{L}(X)$, and $\alpha : \{1, \dots, m\} \rightarrow \{1, \dots, k\}$ be a surjective map for some $k, m \in \mathbb{N}$. Assume the following.*

- (A1_c) *The semigroup $T_m(\cdot)$ is bounded and totally ergodic and $T_j(\cdot)$ has relatively weakly compact orbits for every $1 \leq j \leq m-1$.*
- (A2_c) *Every A_j is compact on the orbits of $T_j(\cdot)$, i.e., $\{A_j T_j(t)x : t \in [0, \infty)\}$ is relatively compact in X for every $x \in X$ and $1 \leq j \leq m-1$.*

Then the entangled ergodic averages

$$\frac{1}{t^k} \int_{[0, t]^k} T_m(s_{\alpha(m)}) A_{m-1} T_{m-1}(s_{\alpha(m-1)}) A_{m-2} \dots A_1 T_1(s_{\alpha(1)}) ds_1 \dots ds_k$$

converge strongly. Denoting the generator of $T_j(\cdot)$ by B_j ($1 \leq j \leq m$), the strong limit is given by the formula

$$\sum_{\substack{\varphi_j \in \sigma_j \ (1 \leq j \leq m) \\ \sum_{i \in \alpha^{-1}(a)} \varphi_i = 0 \ (1 \leq a \leq k)}} P_{\varphi_m}^{(m)} A_{m-1} P_{\varphi_{m-1}}^{(m-1)} A_{m-2} \dots A_1 P_{\varphi_1}^{(1)},$$

where $\sigma_j = P_\sigma(B_j) \cap i\mathbb{R}$ and $P_{\varphi_j}^{(j)}$ is the projection onto the eigenspace of B_j corresponding to $i\varphi_j$, i.e., the mean ergodic projection of the semigroup $(e^{-i\varphi_j t} T_j(t))_{t \geq 0}$.

Proof. Using the arguments from Proposition 2, we may again assume that we have $T_j(\cdot) = T(\cdot)$ and $A_j = A$ for $1 \leq j \leq m-1$. It is to be shown that

$$(16) \quad \frac{1}{t^k} \int_{[0, t]^k} T_m(s_{\alpha(m)}) A T(s_{\alpha(m-1)}) A \dots A T(s_{\alpha(1)}) x ds_1 \dots ds_k$$

converges for every $x \in X$. By Theorem 5, the integrand can be split with the help of the projections P_r and P_s onto X_r and X_s , respectively, and we have

$$\begin{aligned}
& T_m(s_{\alpha(m)})AT(s_{\alpha(m-1)})A \dots AT(s_{\alpha(1)})x \\
&= \sum_{a=1}^{m-1} T_m(s_{\alpha(m)})A \dots AT(s_{\alpha(a)})P_s AT(s_{\alpha(a-1)})P_r A \dots AT(s_{\alpha(1)})P_r x \\
&+ T_m(s_{\alpha(m)})AT(s_{\alpha(m-1)})P_r A \dots AT(s_{\alpha(1)})P_r x.
\end{aligned}$$

The integral means of the second term converge by Theorem 4 to the desired limit, and so it is enough to show that the rest converges in mean to 0, i.e. for every $x \in X$ and $1 \leq a \leq m-1$ one has

$$(17) \quad \lim_{t \rightarrow \infty} \frac{1}{t^k} \int_{[0,t]^k} T_m(s_{\alpha(m)})A \dots AT(s_{\alpha(a)})P_s AT(s_{\alpha(a-1)})P_r A \dots AT(s_{\alpha(1)})P_r x = 0.$$

Consider

$$K := \{AT(s_{a-1})P_r A \dots AT(s_1)P_r x \mid s_{a-1}, \dots, s_1 \in [0, \infty)\}.$$

This set is relatively compact by Lemma 2 and the assumption. As in the discrete case, one can show that the dual space of the smallest $T(\cdot)$ -invariant subspace Y containing K is separable. Note that the separability of Y itself follows from the strong continuity of the semigroup, as it yields a dense countable subset

$$\{AT(s_{a-1})P_r A \dots AT(s_1)P_r x \mid s_{a-1}, \dots, s_1 \in [0, \infty) \cap \mathbb{Q}\}$$

of K .

Theorem 5 then assures the existence of a set $M \subset [0, \infty)$ with density 1 such that

$$\lim_{s_j \in M, s_j \rightarrow \infty} T(s_j)P_s y = 0 \quad \text{weakly for every } y \in K$$

implying $\lim_{s_j \in M, s_j \rightarrow \infty} AT(s_j)P_s y = 0$ weakly for every $y \in K$. Since $\{AT(s)P_s y : s \in [0, \infty)\}$ is relatively compact, and strong convergence in $\mathcal{L}(X)$ implies uniform strong convergence on compact subsets of X , we obtain

$$\lim_{s_j \in M, s_j \rightarrow \infty} \|AT(s_j)P_s y\| = 0 \quad \text{uniformly in } y \in K.$$

Since the set M has density 1, the equation (17) follows from Lemma 3. \square

Note that, as in the discrete case, the class of semigroups satisfying assumption (A1_c) is large including e.g. bounded C_0 -semigroups on reflexive Banach spaces.

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